Analysis of Vibrating Circular Drum

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December 20, 2021

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1 Introduction

The goal of this project was to investigate the initial value problem of a circular membrane, which is given an initial displacement shown in Equation 1, where 'a' is the outer radius of the membrane, 'r' is the radial coordinate, and u_0 is a specified constant.

$$u(r,0) = u_0\left(1 - \frac{r^2}{a^2}\right) \tag{1}$$

The displacement is a function of both radius and time. Also, it is zero at the outer edge (r = a) for all time. The initial velocity of the circular membrane is zero, shown by Equation 2

$$\dot{u}(r,0) = 0 \tag{2}$$

The two-dimensional wave equation describes the vibrations of membranes. In this case, the wave equation is constructed using polar coordinates as shown in Equation 3, where "c" is the radial wave speed.

$$c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) = u_{tt} \tag{3}$$

The first part of this analysis will assume the problem is axisymmetric, eliminating dependence on the θ coordinate from Equation 3 and leading to Equation 4

$$c^2\left(u_{rr} + \frac{1}{r}u_r\right) = u_{tt} \tag{4}$$

2 Separation of variables

To construct the solution to Equation 4, separation of variables was utilized. The solution was separated into a spatial and temporal component as

$$u(r,t) = R(r)T(t) \tag{5}$$

where R(r) is the spatial dependent term and T(t) is the time dependent term. Substituting Equation 5 into Equation 4 gives Equation 6.

$$c^2\left(R''T + \frac{1}{r}R'T\right) = RT'' \tag{6}$$

Dividing Equation 6 by c^2RT and rearranging terms gives the following relation

$$c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) = \frac{T''}{T} = -\lambda^2 \tag{7}$$

where λ must be a constant in order for the equality between spatial and time dependent terms to hold. Thus, two ordinary differential equations can be extracted from Equation 7 and can be expressed as

$$rR'' + R' + r\lambda^2 R = 0 \tag{8}$$

$$T'' + c^2\lambda^2 T = 0 \tag{9}$$

2.1 Spatial Equation

Equation 8 can be identified as a Bessel equation of order 0. However, to develop the solution to Equation 8 two cases must be considered: $\lambda = 0$ and $\lambda \neq 0$. For the case when $\lambda = 0$, Equation 8 becomes

$$rR'' + R' = 0 \tag{10}$$

where this is identified as a Cauchy-Euler equation with repeated roots. This can be readily solved as

$$R(r) = A + Bln(r) \tag{11}$$

Now, considering the case when $\lambda \neq 0$, Equation 8 stays as it is. The general solution can be expressed as

$$R(r) = CJ_0(\lambda r) + DY_0(\lambda r) \tag{12}$$

where J_0 is a Bessel function of the first kind, Y_0 is a Bessel function of the second kind, and C and D are arbitrary constants. Combining Equations 11 and 12 gives the total solution to Equation 8 for both cases as

$$R(r) = \begin{cases} A + Bln(r) & \lambda = 0 \quad (13) \\ CJ_0(\lambda r) + DY_0(\lambda r) & \lambda \neq 0 \quad (14) \end{cases}$$

2.2 Temporal Equation

Equation 9 is a second order linear homogeneous differential equation with constant coefficients. It can be readily solved as

$$T(t) = Ecos(c\lambda t) + Fsin(c\lambda t) \tag{15}$$

where E and F are arbitrary constants. However, Equation 15 is for λ greater than zero. To include the case for $\lambda = 0$, the solution, T(t) can be expressed as

$$T(t) = \begin{cases} Ecos(c\lambda t) + Fsin(c\lambda t) & \lambda \neq 0 \quad (16) \\ Gt + H & \lambda = 0 \quad (17) \end{cases}$$

where again, G and H are arbitrary constants.

3 Boundary Conditions & Initial Conditions

The solution for R(r) in Equations 13 and 14 must be bounded as $r \to 0$. Therefore, the terms that blow up to infinity in Equations 13 and 14 as $r \to 0$ must also go to zero and so $B = D = 0$. Equations 13 and 14 then reduce to

$$R(r) = \begin{cases} A & \lambda = 0 \quad (18) \\ CJ_0(\lambda r) & \lambda \neq 0 \quad (19) \end{cases}$$

Now, the displacement of the circular membrane on the boundary (r=a) is zero, therefore Equations 18 and 19 are set to zero at r=a.

$$R(a) = 0 = \begin{cases} A & \lambda = 0 \quad (20) \\ CJ_0(\lambda a) & \lambda \neq 0 \quad (21) \end{cases}$$

From Equations 20 and 21 it can be deduced that A = 0 and $J_0(\lambda a) = 0$. However, for the Bessel function, $J_0(\lambda a)$ to be zero, λa must be one of the roots of the Bessel function. Therefore

$$\lambda_n = \frac{Z_n}{a} \tag{22}$$

where Z_n are the n roots of the Bessel function of the first kind of order 0. The solution R(r) can then be expressed as

$$R(r) = CJ_0(\frac{Z_n}{a}r) \tag{23}$$

Now, the circular membrane has the initial conditions given by Equations 1 and 2. Applying the initial velocity condition to Equations 16 and 17 gives the following relation

$$T'(0) = 0 = \begin{cases} \frac{F}{c\lambda} & \lambda \neq 0 \quad (24) \\ G & \lambda = 0 \quad (25) \end{cases}$$

therefore, the temporal component reduces to

$$T(t) = \begin{cases} Ecos(c\lambda t) & \lambda \neq 0 \quad (26) \\ H & \lambda = 0 \quad (27) \end{cases}$$

Using the superposition of solutions and the results from Equations 23, 26, and 27, the solution u(r,t) can be expressed as

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\frac{Z_n}{a}r) cos(c\frac{Z_n}{a}t) \tag{28}$$

Note, the initial displacement condition was not used yet, it will now be used to determine the coefficients of the series. The initial displacement gives the following relation as

$$u(r,0) = u_0 \left(1 - \frac{r^2}{a^2}\right) = \sum_{n=1}^{\infty} A_n J_0(\frac{Z_n}{a}r) \tag{29}$$

where the coefficients, A_n, can now be determined by evaluating the following integral.

$$A_n = \frac{2}{a^2[J_1(Z_n)]^2} \int_0^a u_0 \left(1 - \frac{r^2}{a^2}\right) J_0(\frac{Z_n}{a}r) r\, dr \tag{30}$$

To solve for A_n, let $Q = \frac{Z_n}{a}r$ and make a change of variables. Equation 30 becomes

$$A_n = \frac{2}{Z_n^4[J_1(Z_n)]^2} \int_0^{Z_n} u_0 \left(Z_n^2 - Q^2\right) J_0(Q) Q\, dQ \tag{31}$$

$$A_n = \frac{2u_0}{Z_n^4[J_1(Z_n)]^2} \int_0^{Z_n} \left(Z_n^2 - Q^2\right) \frac{d}{dQ}[J_1(Q)Q]\, dQ \tag{32}$$

where Equation 32 was constructed using Bessel function properties. Using integration by parts re-expresses A_n as

$$A_n = \frac{2u_0}{Z_n^4[J_1(Z_n)]^2} \left[(z_n^2 - Q^2) J_1(Q) Q \Big|_0^{Z_n} + \int_0^{Z_n} 2Q J_1(Q) Q\, dQ \right] \tag{33}$$

The first term in the brackets in Equation 33 goes to zero, reducing the expression of $\mathrm{A_n}$ to

$$A_n = \frac{2u_0}{Z_n^4[J_1(Z_n)]^2}\int_0^{Z_n} 2QJ_1(Q)Q\,dQ \tag{34}$$

$$A_n = \frac{4u_0}{Z_n^2[J_1(Z_n)]^2}J_2(Z_n) \tag{35}$$

$$A_n = \frac{8u_0}{Z_n^3 J_1(Z_n)} \tag{36}$$

where Bessel function properties were used to simplify Equations 35 and 36. The final solution for the vibrations of the circular membrane can then be expressed as

$$u(r,t) = \sum_{n=1}^{\infty}\frac{8u_0}{Z_n^3 J_1(Z_n)}J_0(\frac{Z_n}{a}r)cos(c\frac{Z_n}{a}t) \tag{37}$$

where $\mathrm{Z_n}$ are the roots of the zero-order Bessel function of the first kind.

4 Orthogonality of Eigenfunctions

From the preceding analysis, the eigenfunctions were determined to be $J_0(\frac{Z_n}{a}r)$ and the corresponding eigenfrequecies were $\frac{Z_n}{a}$. The question remains whether the eigenfunctions are orthogonal for distinct values of n. To prove this, the following relation must be established

$$\int_0^a rJ_0(\frac{Z_m}{a}r)J_0(\frac{Z_n}{a}r)dr = 0 \tag{38}$$

where m and n are roots of the Bessel function in question but m $\neq$ n. Bessel functions of the first kind, $J_v(r)$, satisfy the equation

$$r_2R'' + rR' + (r^2 - v^2)R = 0 \tag{39}$$

where v designates the order of the Bessel equation. Equation 39 can be rewritten as

$$r(rR')' + (r^2 - v^2)R = 0 \tag{40}$$

Now consider two Bessel function solutions, $J_v(\frac{m}{a}r)$ and $J_v(\frac{n}{a}r)$ where m $\neq$ n. These solutions satisfy the following two equations respectively.

$$r(rR')' + (\frac{m^2r^2}{a^2} - v^2)R = 0 \tag{41}$$

$$r(rR')' + (\frac{n^2r^2}{a^2} - v^2)R = 0 \tag{42}$$

By multiplying Equation 41 with $J_v(\frac{n}{a}r)$ and Equation 42 with $J_v(\frac{m}{a}r)$, subtracting the two expressions and then dividing by r, the following expression can be deduced

$$\frac{d}{dr}\left[rJ_v(\frac{n}{a}r)J_v'(\frac{m}{a}r) - rJ_v(\frac{m}{a}r)J_v'(\frac{n}{a}r)\right] + \frac{m^2-n^2}{a^2}rJ_v(\frac{m}{a}r)J_v(\frac{n}{a}r) = 0 \tag{43}$$

Integrating Equation 43 over $0 \rightarrow$ a produces the expression

$$rJ_v(\frac{n}{a}r)J_v'(\frac{m}{a}r) - rJ_v(\frac{m}{a}r)J_v'(\frac{n}{a}r)\Big|_0^a + \frac{m^2 - n^2}{a^2}\int_0^a rJ_v(\frac{m}{a}r)J_v(\frac{n}{a}r) = 0 \tag{44}$$

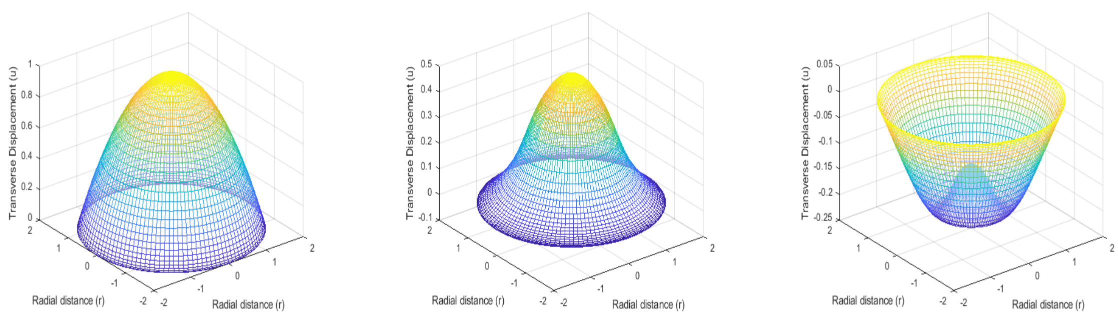
The first term in Equation 44 goes to zero because at a $J_v(n) = J_v(m) = 0$ and at zero r = 0. Therefore, the second term is the only term that remains and since m $\neq$ n the integral must be zero, thus

$$\int_0^a rJ_v(\frac{m}{a}r)J_v(\frac{n}{a}r) = 0 \tag{45}$$

and orthogonality of the eigenfunctions is demonstrated.

5 Convergence of Solution

The solution to the circular membrane problem was determined by Equation 37. Using MATLAB, displacement surfaces were plotted at different times with $u_0 = 1$, c = 1, a = 2, and 8 terms used in the series (Figure 1).



(a) Solution at Time = 0

(b) Solution at Time = 1

(c) Solution at Time = 1.5

Figure 1: Solution at times t = 0, 1, 1.5

To investigate the convergence of the solution, the L2 norm of the solution set was determined as each term was added to the series. From the plot in Figure 2 it can be seen that as terms are added, the initially the curve dips down, but then flattens out as more terms are added. This flattening demonstrates the convergence of the solution.

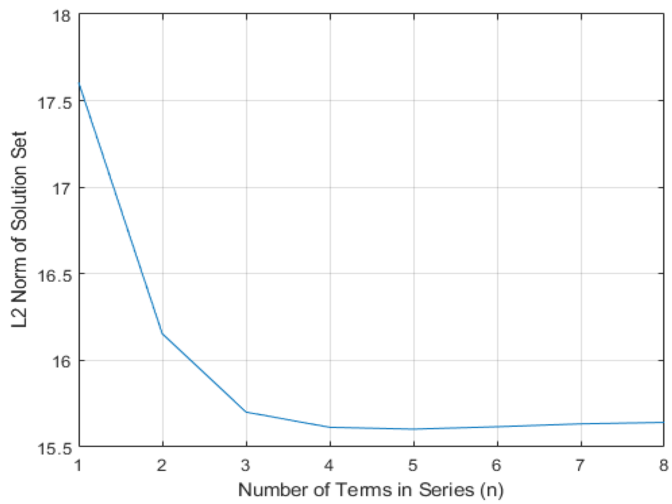


Figure 2: Convergence history of membrane wave solution.

6 Modifications for the Non-axisymmetric Case

In the case where the initial displacement is not asymmetric, the θ dependence cannot be neglected in the two-dimensional wave from Equation 3. With that said, instead of getting two ordinary differential equations from the process of separation of variables, there would be three. The solution would be assumed to be separable in the form

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t) \tag{46}$$

A similar procedure as presented in the previous analysis would be followed. The radial and angular analyses would result in the Legendre Equation, rather than a Bessel Equation. This is because of the $\frac{1}{r^2}$ term in the second order θ derivative of Equation 3. The temporal analysis would remain the same. Then, because Legendre Functions have the same orthogonality properties as Bessel Functions, the orthogonality of eigenfunctions would be proven in the same way.

7 References

[1] Greenberg, M. D., 1998, *Advanced Engineering Mathematics*, Prentice-Hall, Upper Saddle River (New Jersey).