Analysis of the Duffing Equation

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Duffing Equation

In this analysis, the nonlinear ODE named the Duffing Equation was investigated both analytically and numerically. The general form of the Duffing Equation can be expressed as

$$m\ddot{u} + r\dot{u} + \alpha u + \beta u^3 = F_0 cos\Omega t \tag{1}$$

where m, r, α, β, F_0, and Ω are arbitrary constants, u is the dependent variable, and t is the independent temporal variable. In the physical spring-mass system, the variable u represents displacement, α and β make up the nonlinear spring constant, r is a damping coefficient, and m is the mass. The right side of Equation 1 is the forcing term of the nonlinear differential equation, which oscillates in time with frequency Ω.

Equilibrium Points

It is important to investigate the equilibrium points of this nonlinear differential equation and this can be done analytically by considering the homogeneous case given by Equation 2.

$$m\ddot{u} + r\dot{u} + \alpha u + \beta u^3 = 0 \tag{2}$$

Defining the state variables of Equation 2 as

$$u = u \tag{3}$$

$$v = \dot{u} \tag{4}$$

and taking the time derivative provides the state space representation of Equations 3 and 4.

$$\dot{u} = v \tag{5}$$

$$\dot{v} = \frac{-\mathrm{r}v - \alpha\mathrm{u} - \beta\mathrm{u}^3}{m} \tag{6}$$

Substituting values for the constants as m = r = β = 1 and α = -1, Equation 5 and 6 can be rewritten as

$$\dot{u} = v \tag{7}$$

$$\dot{v} = -v + \mathrm{u} - \mathrm{u}^3 \tag{8}$$

Setting Equations 7 and 8 equal to zero and solving for u and v gives

$$v = 0 \tag{9}$$

$$\mathrm{u} = 0, \pm 1 \tag{10}$$

therefore, the equilibrium points for the system are (0,0), (1,0), and (-1,0). To investigate the stability of the equilibrium points, Equations 7 and 8 must be linearized about the equilibrium points. Linearization in terms of the two variables, u and v, can be done using Taylor series expansion. Equations 7 and 8 can be rewritten as

$$\dot{u} = v = P(u, v) \tag{11}$$

$$\dot{v} = -v + \mathrm{u} - \mathrm{u}^3 = Q(u, v) \tag{12}$$

and linearization about the equilibrium points approximates Equations 11 and 12 as

$$\dot{u} = P_u(u, v)(u - u_s) + P_v(u, v)(v - v_s) \tag{13}$$

$$\dot{v} = Q_u(u, v)(u - u_s) + Q_v(u, v)(v - v_s) \tag{14}$$

where P_u and P_v are the partial derivatives of P(u,v) with respect to u and v respectively, Q_u and Q_v are the partial derivatives of Q(u,v) with respect to u and v respectively, and u_s and v_s are the u and v components of the equilibrium point of interest. The pre-factors in Equations 13 and 14 can be expressed as

$$P_u(u, v) = 0 \tag{15}$$

$$P_v(u, v) = 1 \tag{16}$$

$$Q_u(u, v) = 1 - 3u_s^2 \tag{17}$$

$$Q_v(u, v) = -1 \tag{18}$$

where Q_u depends on the particular equilibrium point.

Equilibrium Point (0,0)

Equations 13 and 14 can be linearized about the equilibrium point at the origin and expressed as

$$\dot{u} = v \tag{19}$$

$$\dot{v} = u - v \tag{20}$$

The roots of the characteristic equation can be evaluated as

$$\lambda_{1,2} = \frac{(a + d) \pm \sqrt{(a - d)^2 + 4bc}}{2} \tag{21}$$

and in this case a = 0, b = 1, c = 1, and d = -1; with these coefficient values, the roots can be expressed as

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2} \tag{22}$$

where the roots are real and have opposite signs implying that there is a saddle at the origin.

Equilibrium Points (±1,0)

For the equilibrium point (1,0), Equations 13 and 14 can be linearized as

$$\dot{u} = v \tag{23}$$

$$\dot{v} = -2(u-1) - v \tag{24}$$

By defining $V = v\text{-}v_s$ and $U = u\text{-}u_s$, Equations 23 and 24 can be rewritten as

$$\dot{U} = V \tag{25}$$

$$\dot{V} = -2U - V \tag{26}$$

and the roots can be identified as

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2} \tag{27}$$

which represent a complex conjugate pair with negative real roots, this implies that the point (1,0) is a stable focus. Similarly, for the point (-1,0), the system can be linearized and expressed as

$$\dot{u} = v \tag{28}$$

$$\dot{v} = -2(u-(-1)) - v \tag{29}$$

And using the previous variable substitutions to shift the point of interest to the origin, the Equation 25 and 26 are recovered, implying that there is also a stable focus at the point (-1,0).

Phase Portraits and Parameter Node Bifurcation

The phase diagram for the unforced system with $\alpha = -1$ and $r = m = \beta = 1$ was plotted in MATLAB, and is shown in Figure 1. It can be seen from the Figure that there is indeed a stable focus at the points (1,0) and (-1,0) as well as a saddle at the origin.

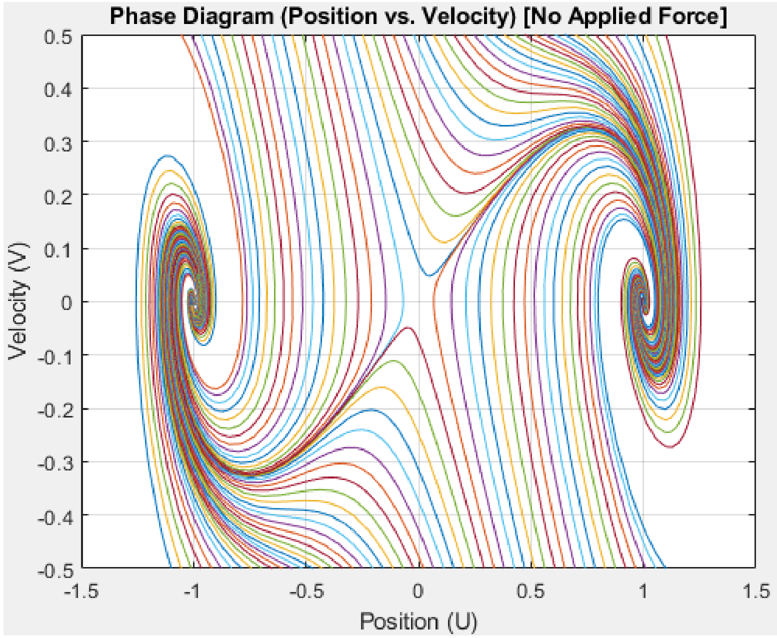

Figure 1. Phase diagram of unforced system with parameter values α = -1 and r = m = β = 1

An interesting area to explore is how the equilibrium points, or nodes, evolve as we vary the parameters of the homogenous equation. Figure 2 includes the case of r = m = β = α = 1, which has only a single stable node at (0,0).

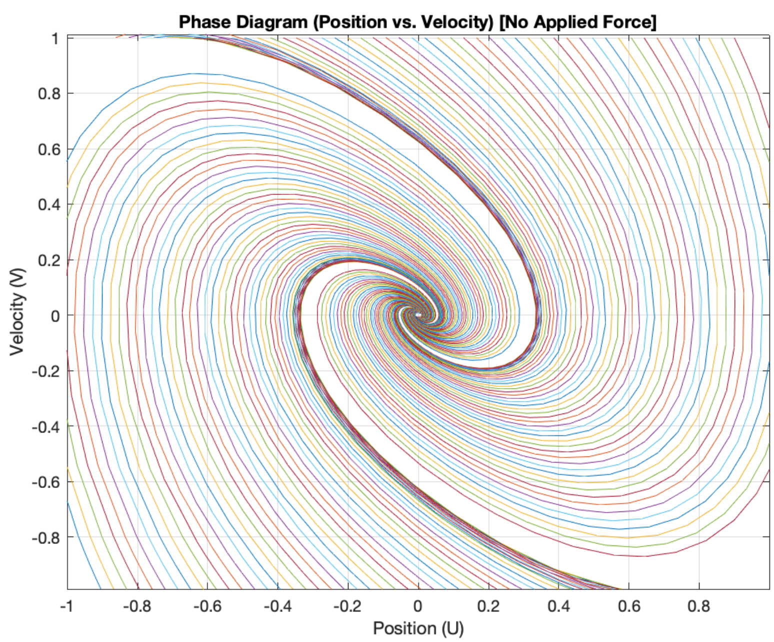

Figure 2. Phase diagram of unforced system with parameter values α = r = m = β = 1

When the parameters are adjusted to β = -1 and r = m = α = 1, we again find three nodes. They are in the same locations as the nodes in Figure 1, but the stability is reversed. A stable node is located at (0,0) and two saddle nodes are located at (1,0) and (-1,0). The shift from one node to three corresponding to α and β switching sign can be visualized with a bifurcation diagram. Figure 3 shows this bifurcation, with the previously used α values, as well as the node locations drawn.

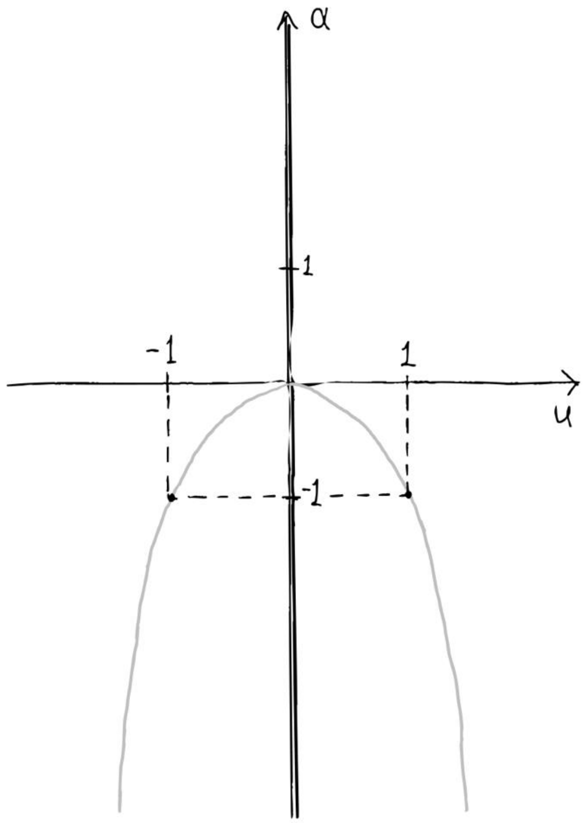

Figure 3. Bifurcation diagram of α parameter nodes. For α > 0 there is a single stable spiral. For α < 0 the node bifurcates into 2 stable spirals and 1 saddle.

Now considering the physical system along with the current understanding, we investigate the effect of the forcing term. It will cause phase space trajectories to be shifted by some Δu through kinematics. Then, if the shift is sufficiently large, those trajectories will “jump” to the next stable node in the direction of the shift. This is proven mathematically when the forcing term is given a nonzero value in the numerical simulation.

Numerical Simulation with Nonzero Forcing Term

The homogeneous Duffing Equation was used to investigate the equilibrium points, but it is also important to understand how the system behaves under an applied force. The system was numerically solved in MATLAB by defining the amplitude of the force as $F_0 = 1$, $\alpha = -1$, and keeping all of the other constants equal to 1. The initial conditions were chosen so that the position (U) and velocity (V) of the system were zero. A phase diagram was generated from the numerical simulation and is shown in Figure 4. Based on the chosen constants, as time approaches infinity, the system approaches a limit cycle that jumps back and forth between the equilibrium points (-1,0) and (1,0).

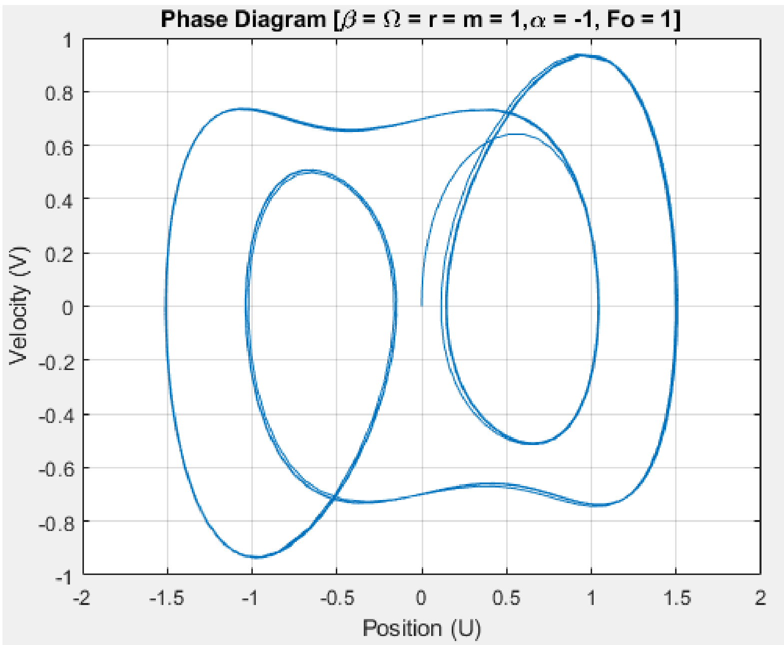

Figure 4. Phase diagram for forced system resulting in limit cycle

The origin is clearly avoided which makes sense, since there is a saddle there. The limit cycle shown in Figure 4 can be visualized further by looking at the numerical solution to the Duffing Equation over time shown in Figure 5. There is some distortion in the periodicity at the beginning, but as time goes on, a well-defined periodicity starts to show up in the time response.

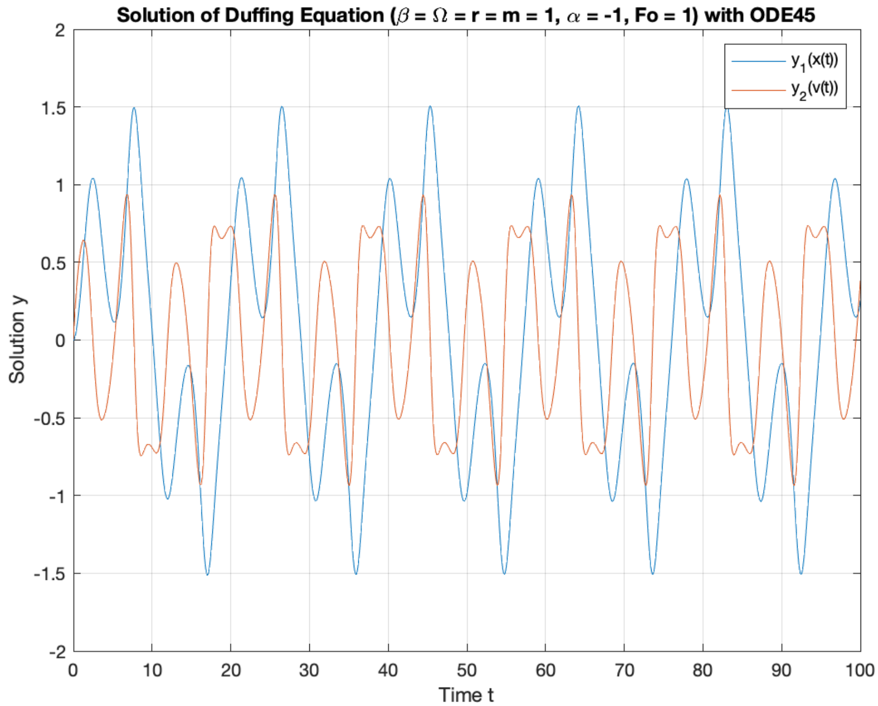

Figure 5. Time response of forced duffing equation with $F_0 = \Omega = m = r = \beta = 1$ and $\alpha = -1$.

The Chaotic System

Adjusting the parameter values of the Duffing Equation to specific ranges can lead to a chaotic system. This idea is investigated by using the parameter values given by $\beta = m = 1$, $r = 0.3$, $\alpha = -1$, $\Omega = 1.2$, and $F_0 = 0.5$ [1]. The initial conditions are given 3 different values, shown in Figures 6 – 8. From the mathematical definition of chaos, the system should have a different solution for each of these sets of initial conditions.

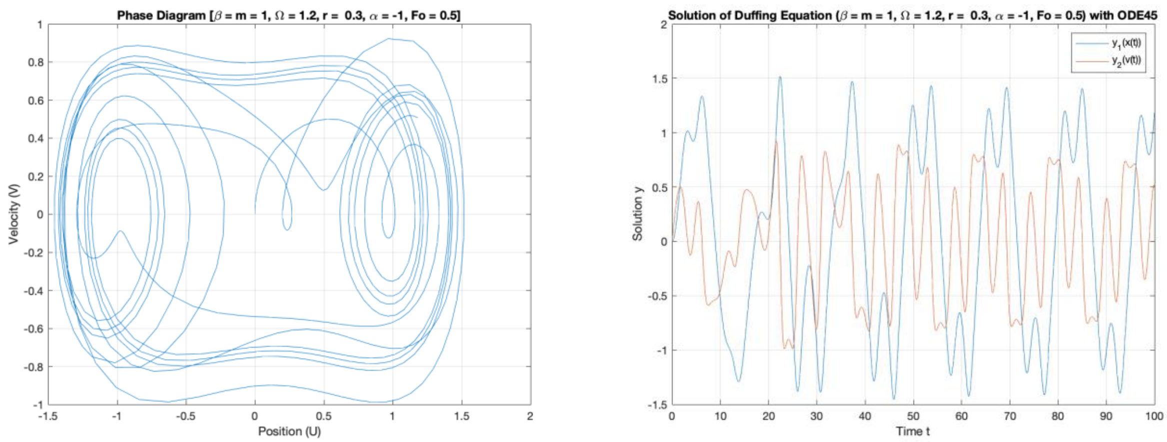

Figure 6. Phase diagram (left) and time response plot (right) for forced duffing equation with β = m = 1, r = 0.3, α = -1, Ω = 1.2, and F_0 = 0.5 (Initial conditions: (U,V) = (0,0))

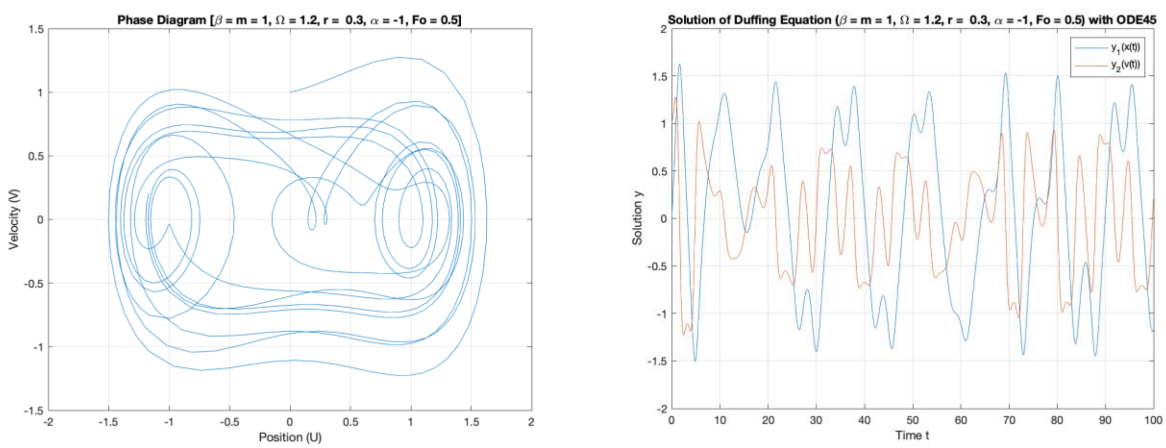

Figure 7. Phase diagram (left) and time response plot (right) for forced duffing equation with β = m = 1, r = 0.3, α = -1, Ω = 1.2, and F_0 = 0.5 (Initial conditions: (U,V) = (0,1))

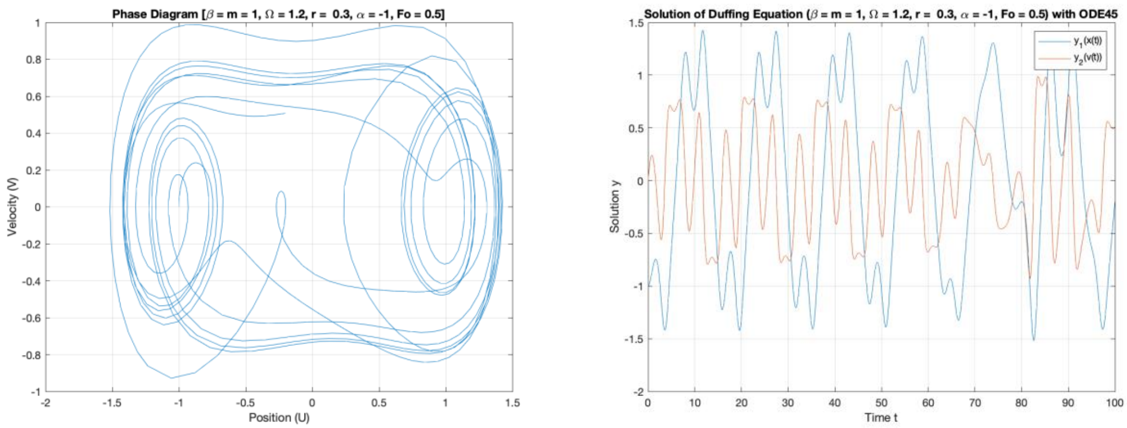

Figure 8. Phase diagram (left) and time response plot (right) for forced duffing equation with β = m = 1, r = 0.3, α = -1, Ω = 1.2, and F_0 = 0.5 (Initial conditions: (U,V) = (-1,0))

From the above images, it is clear that different initial conditions can create significantly different solutions to the Duffing equation. This very unique property can only be found in nonlinear systems such as this [2]. We investigated both the homogenous and forced Duffing equation in this work. The Duffing equation provides a great example of how a mathematical concept such as chaos can be demonstrated in a physical system.

References

[1] Greenberg, M. D., 1998, *Advanced Engineering Mathematics*, Prentice-Hall, Upper Saddle River (New Jersey).

[2] Strogatz, S. H., 2015, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*.