Analysis of Symmetric Spinning Top

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September 2023

1 Introduction

This analysis is focused on the dynamics of a symmetric spinning top, with one end fixed at the origin. Figure 1 shows a depiction of the system under consideration.

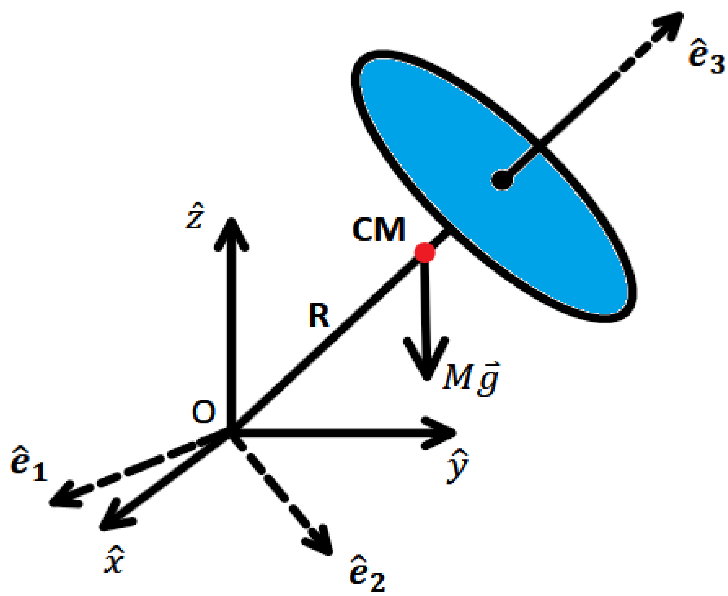

Figure 1: Diagram of Symmetric Spinning Top

A few things to consider in this problem:

- The space frame is designated by $\hat{x}$, $\hat{y}$, and $\hat{z}$
- Friction is neglected.
- The principal axes of the symmetric spinning top (also called the body axes in this analysis) are designated as $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ where the third is the vector along the spinning axis and the others are free (due to symmetry) as long as they are orthogonal to the third and to each other.

2 Derivation of Lagrangian

In order to analyze the dynamics, let's derive the Lagrangian for the system using Euler angles . Consider the convention below:

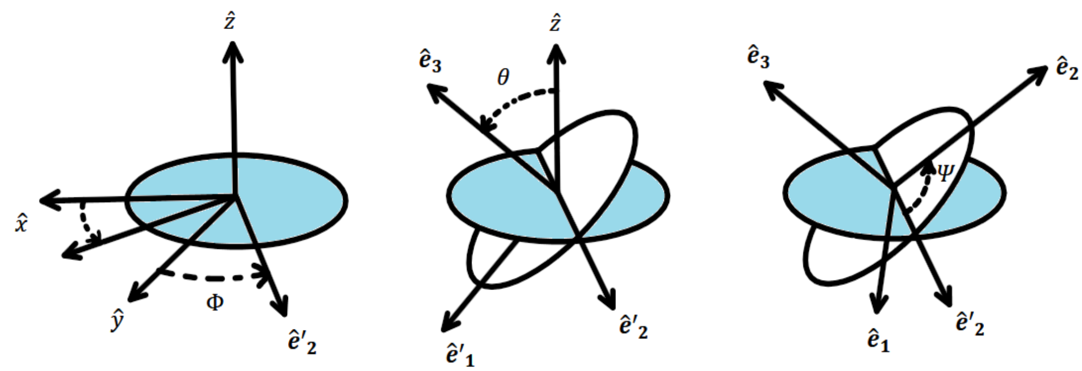

Figure 2: The axes $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ are initially coincident with the stationary frame. After three successive rotations, the body axes can be transformed into any orientation. **Left**: The first rotation is ϕ about axis $\hat{\boldsymbol{z}}$. **Center**: The second rotation is θ about axis $\hat{\boldsymbol{e}}_2'$. **Right**: The third rotation is ψ about axis $\hat{\boldsymbol{e}}_3$ [1].

With the Euler angle convention outlined above, we can define the angular velocity of the body frame with respect to the space frame as:

$$\boldsymbol{\omega} = \dot{\phi}\hat{\boldsymbol{z}} + \dot{\theta}\hat{\boldsymbol{e}}_2' + \dot{\psi}\hat{\boldsymbol{e}}_3 \tag{1}$$

Now, recall that the principal axes are the eigenvectors in the eigenvalue problem

$$\boldsymbol{I}\boldsymbol{\omega} = \lambda\boldsymbol{\omega} \tag{2}$$

where $\boldsymbol{I}$ is the inertia tensor, $\boldsymbol{\omega}$ is the angular velocity vector (in this case eigenvector), and λ is the corresponding eigenvalue (in this case the corresponding principal moment). In the scenario where two principal moments are equal, the corresponding two principal axes are free to take any direction that is orthogonal to the third as well as each other. For the symmetric top, the two principal moments, λ_1 and λ_2, are equal. Therefore, we can pick $\hat{\boldsymbol{e}}_1'$ and $\hat{\boldsymbol{e}}_2'$ as the first two principal axes; Using this fact and the relation

$$\hat{\boldsymbol{z}} = (cos\theta)\hat{\boldsymbol{e}}_3 - (sin\theta)\hat{\boldsymbol{e}}_1' \tag{3}$$

Equation 1 can easily be rewritten with respect to the body frame using

$$\boldsymbol{\omega} = (-\dot{\phi}sin\theta)\hat{\boldsymbol{e}}_1' + \dot{\theta}\hat{\boldsymbol{e}}_2' + (\dot{\psi} + \dot{\phi}cos\theta)\hat{\boldsymbol{e}}_3 \tag{4}$$

The Lagrangian demands a kinetic energy term, to obtain this we need the angular momentum vector $\boldsymbol{L}$. Generally, the angular momentum can be expressed as

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} \tag{5}$$

Since we are using a set of principal axes, the inertia tensor is diagonal, with the principal moments along the diagonal. With this fact, and Equation 4, the angular momentum can be expressed as

$$\boldsymbol{L} = (-\lambda_1 \dot{\phi} sin\theta)\hat{\boldsymbol{e}}_1' + \lambda_1 \dot{\theta} \hat{\boldsymbol{e}}_2' + \lambda_3(\dot{\psi} + \dot{\phi} cos\theta)\hat{\boldsymbol{e}}_3 \tag{6}$$

For future reference, let's define the following, first of which is the momentum about the body axis $\boldsymbol{e_3}$:

$$L_3 = \lambda_3 \omega_3 = \lambda_3(\dot{\psi} + \dot{\phi} cos\theta) \tag{7}$$

and the second is the momentum about the z-axis of the space frame:

$$L_z = \boldsymbol{L} \cdot \hat{\boldsymbol{z}} \tag{8}$$

$$L_z = ((-\lambda_1 \dot{\phi} sin\theta)\boldsymbol{e_1'} + \lambda_1 \dot{\theta} \boldsymbol{e_2'} + \lambda_3(\dot{\psi} + \dot{\phi} cos\theta)\boldsymbol{e_3}) \cdot ((cos\theta)\boldsymbol{e_3} - (sin\theta)\boldsymbol{e_1'}) \tag{9}$$

$$L_z = \lambda_1 \dot{\phi} sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi} cos\theta)cos\theta \tag{10}$$

$$L_z = \lambda_1 \dot{\phi} sin^2\theta + L_3 cos\theta \tag{11}$$

Now we can write the kinetic energy using:

$$T = \frac{1}{2}\boldsymbol{L} \cdot \boldsymbol{\omega} \tag{12}$$

That gives us

$$T = \frac{1}{2}\lambda_1(\dot{\phi}^2 sin^2\theta + \dot{\theta})^2 + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi} cos\theta)^2 \tag{13}$$

The potential energy of the rigid body can be defined as

$$U = MgRcos\theta \tag{14}$$

where M is the total mass, g is the acceleration due to gravity, and R is the distance between the pivot point O and the center of mass. Thus, the Lagrangian of the system can be expressed as:

$$\mathcal{L} = T - U \tag{15}$$

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi} cos\theta)^2 - MgRcos\theta \tag{16}$$

3 Euler-Lagrange equations

The general form of the Euler-Lagrange equations, without generalized forces, for a system with n generalized coordinates q_i and corresponding velocities $\dot{q}_i$ is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{17}$$

The three resulting Lagrange equations are as follows

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{19}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{20}$$

3.1 θ Equation

Using the Lagrangian in Equation 5, Equation 18 gives

$$\lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2 sin\theta cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}cos\theta)\dot{\phi}sin\theta + MgRsin\theta \tag{21}$$

3.2 ϕ Equation

ϕ does not appear in the Lagrangian, thus:

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}cos\theta)cos\theta = const \tag{22}$$

This implies that

$$L_z = const \tag{23}$$

3.3 ψ Equation

ψ does not appear in the Lagrangian, thus:

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}cos\theta) = const \tag{24}$$

This implies that

$$L_3 = const \tag{25}$$

3.4 E-L Notes

The Euler-Lagrange equations have revealed that L_z and L_3 are constant. These constants and Equation 21 fully provide the dynamics of the system. A numerical evaluation of these dynamics is discussed in the next section.

4 Numerical Evaluation and Simulation

Let's solve this system numerically for θ, ϕ, and ψ . To do so we begin by setting up the system of equations. Equation 21 can be rewritten as

$$\ddot{\theta} = \dot{\phi}^2 sin\theta cos\theta - \frac{\lambda_3}{\lambda_1}(\dot{\psi} + \dot{\phi}cos\theta)\dot{\phi}sin\theta + \frac{1}{\lambda_1}MgRsin\theta \tag{26}$$

and from Equation11 we get

$$\dot{\phi} = \frac{L_z - L_3 cos\theta}{\lambda_1 sin^2\theta} \tag{27}$$

and from Equation 22 we get

$$\dot{\psi} = \frac{L_3}{\lambda_3} - \dot{\phi}cos\theta \tag{28}$$

Equations 26-28 can be used the construct a set of four first order differential equations which are expressed as

$$\frac{d\theta}{dt} = \dot{\theta} \tag{29}$$

$$\frac{d\dot{\theta}}{dt} = \dot{\phi}^2 sin\theta cos\theta - \frac{\lambda_3}{\lambda_1}(\dot{\psi} + \dot{\phi}cos\theta)\dot{\phi}sin\theta + \frac{1}{\lambda_1}MgRsin\theta \tag{30}$$

$$\frac{d\phi}{dt} = \frac{L_z - L_3 cos\theta}{\lambda_1 sin^2\theta} \tag{31}$$

$$\frac{d\psi}{dt} = \frac{L_3}{\lambda_3} - \dot{\phi}cos\theta \tag{32}$$

Equations 29-32 can be solved simultaneously using Runga Kutta Methods. In this case, MATLAB's ODE45 function was used to solve the system.

4.1 Example Case

Say you were to take hold of the tip of a spinning top that was fixed at the other end, but still allowed to pivot about that point, and you spun it with a torque that was purely in the $\hat{\boldsymbol{e}}_3$ direction. Now if we assume that the top had no other angular momentum except that imparted by the spinner then $L_z = L_3 cos\theta_0$. Although this is the typical case of spinning a top, it results in an interesting result. Recall Equation 27, if we look at the initial case where $\theta = \theta_0$ we have

$$\dot{\phi}_0 = \frac{L_z - L_3 cos\theta_0}{\lambda_1 sin^2\theta_0} \tag{33}$$

But we said in this specific case $L_z = L_3 cos\theta_0$ and so $\dot{\phi}_0 = 0$. That means any time the system returns to the initial θ_0 the angular velocity about the $\hat{\boldsymbol{z}}$ axis is zero. This effectively results in a periodic cusp trajectory, see Figure 3. Note, once the top is initially spun the L_z and L_3 values stay constant for all time. A MATLAB script is attached which shows an animation of this case, parameters can be adjusted to explore other cases as well.

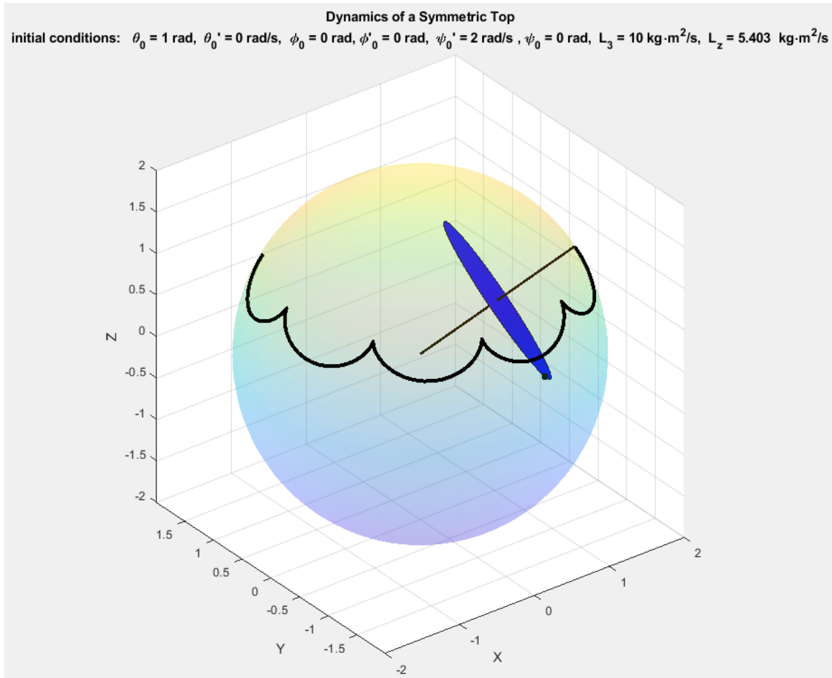

Figure 3: Simulation

References

[1] John R. Taylor. *Classical mechanics /*. University Science Books,, Sausalito, Calif. :, c2005.