N-Link Pendulum Dynamics

Scott Nguyen

April 2024

1 Introduction

This document showcases a dynamical analysis of an N-link pendulum using Euler-Lagrange equations. To begin, let's consider a 2D pendulum with N links and N joints depicted in Figure 1. This pendulum is subjected to gravity in the $-\hat{y}$ direction and dampening frictional forces at the joints.

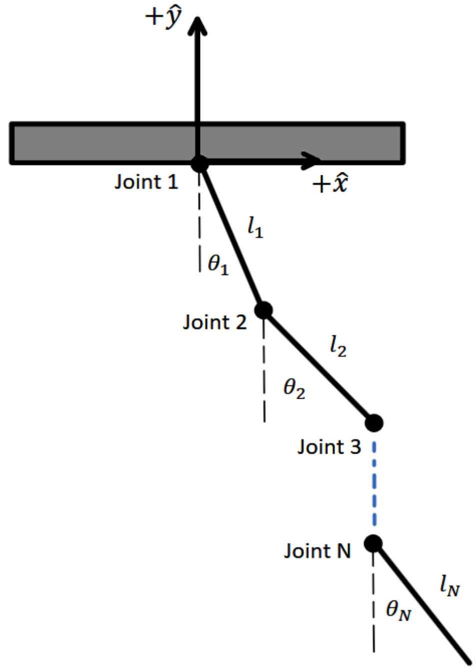

Figure 1: 2D N-link Pendulum

To analyze the dynamics of this problem we use the Euler-Lagrange equations with with Rayleigh's Dissipation Function to model the non-conservative frictional forces. For n generalized coordinates, the Euler Lagrange equations take the general form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \tag{1}$$

where $\mathcal{L}$ is the Lagrangian of the system q_i are the generalized coordinates, $\dot{q}_i$ are the generalized velocities, and Q_i are the generalized forces.

1.1 The Lagrangian & Non-conservative Forces

To model the non-conservative frictional forces at the joints we will use Rayleigh's Dissipation Function. Recall that Rayleigh's Dissipation Function in 2D Cartesian coordinates can be expressed as

$$\mathcal{F} = \frac{1}{2}\sum_{j=1}(k_{jx}v_{jx}^2 + k_{jy}v_{jy}^2) \tag{2}$$

where j denotes the j^{th} particle, $k_{j(dir)}$ denotes the coefficient of friction associated with a particle j and direction, and $v_{j(dir)}$ denotes the velocity of a particle j in a particular direction. Without going through the entire derivation, it can be shown that the component of the generalized force resulting from the force of friction can be expressed as

$$Q_i = -\frac{\partial \mathcal{F}}{\partial \dot{q}_i} \tag{3}$$

Based on this contribution from friction, Equation 1 can be expressed as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0 \tag{4}$$

For the N-link pendulum problem the generalized coordinates are the the pendulum angles, this can be represented as

$$q_i = \{\theta_1, \theta_2, ...\theta_N\} \tag{5}$$

where the generalized velocities are

$$q_i = \{\dot{\theta}_1, \dot{\theta}_2, ...\dot{\theta}_1\} \tag{6}$$

That means the Euler-Lagrange Equations can be written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}_i} = 0 \tag{7}$$

Now, let's assume the the joints are subjected to rotational viscous damping and that

$$k_j = k_{jx} = k_{jy} \tag{8}$$

Therefore, Equation 2 can be rewritten as

$$\mathcal{F} = \frac{1}{2}\sum_{j=1}(k_j\dot{\theta}_j^2) \tag{9}$$

So far we have a valid set of Euler-Lagrange Equations with a generalized force component from friction, now let's define the Lagrangian for the system. For this problem we are considering the rods that connect particles to be mass less and that the particles have mass m_i. The x and y coordinates for each particle, in terms of the generalized coordinates, are given by

$$x_i = l_i sin\theta_i + l_{i-1} sin\theta_{i-1} ... + l_1 sin\theta_1 \tag{10}$$

$$y_i = -l_i cos\theta_i - l_{i-1} cos\theta_{i-1} ... - l_1 cos\theta_1 \tag{11}$$

We shall stick to using x_i and y_i to keep the Lagrangian compact but understand that that it is just a shorthand for the right hand side of the expressions above. Now, differentiating these expressions with respect to time, the kinetic energy can be expressed as

$$KE = \frac{1}{2} \sum_{i=1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2) \tag{12}$$

Since gravity is pointing in the $-\hat{y}$ direction, the potential energy can be expressed as

$$PE = \sum_{i=1}^{N} (m_i g y_i) \tag{13}$$

With Equations 12 and 13 the Lagrangian can be expressed as

$$L = \sum_{i=1}^{N} \frac{1}{2} m_i(\dot{x}_i^2 + \dot{y}_i^2) - (m_i g y_i) \tag{14}$$

Substituting the Lagrangian into Equation 7 provides N-equations of motion. These equations are a set of nonlinear coupled differential equations.

1.2 Solving the Equations of Motion Numerically

To solve the equations of motion numerically, we need to isolate the generalized accelerations in terms of the generalized coordinates and velocities. To do so, consider the set of Euler-Lagrange equations:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}_1} &= 0 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}_2} &= 0 \\
&\vdots \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_N}\right) - \frac{\partial \mathcal{L}}{\partial \theta_N} + \frac{\partial \mathcal{F}}{\partial \dot{\theta}_N} &= 0
\end{aligned}$$

Each of these equations are functions of every generalized coordinate, velocity, and acceleration, therefore once they are evaluated they take the following form

$$f_1(\theta_i, \dot{\theta}_i, \ddot{\theta}_i) = 0$$
$$f_2(\theta_i, \dot{\theta}_i, \ddot{\theta}_i) = 0$$
$$\vdots$$
$$f_N(\theta_i, \dot{\theta}_i, \ddot{\theta}_i) = 0$$

Now, there is a way to untangle this set of equations such as to isolate the accelerations in terms of the generalized coordinates and velocity. Start with first equation in the set, isolate $\ddot{\theta}_1$, then substitute $\ddot{\theta}_1$ into the second equation such that f_2 is no longer an explicit function of $\ddot{\theta}_1$. Isolate $\ddot{\theta}_2$ in the second equation and substitute $\ddot{\theta}_1$ into the third equation first, then substitute $\ddot{\theta}_2$ into the third equation. The algorithm continues by isolating accelerations of the current equation and substituting previous accelerations successively starting from the first. The result is that the last equation in the set will be only a function of the generalized coordinates and velocities, after which a backward substitution can be invoked to express every acceleration in this way. An example of the forward substitution is shown in Figure 2 and the backward substitution is shown in Figure 3, both considering 4 links of a chain.

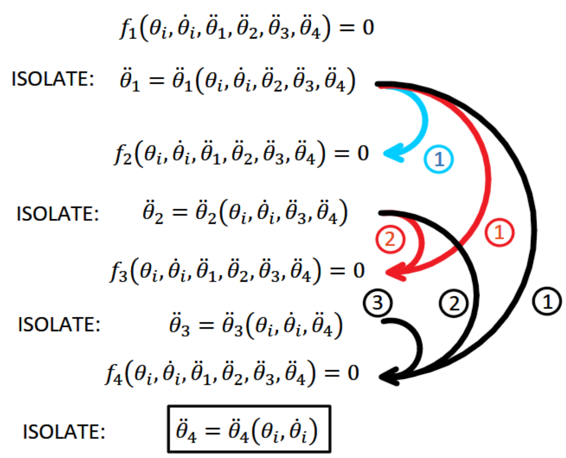

Figure 2: Forward Substitution for 4 links. The bubble numbers indicate what sequence of substitution should take place. The result is the fourth generalized acceleration being expressed solely in terms of generalized coordinates and velocities

$$
\begin{aligned}
\ddot{\theta}_4 &= \boxed{\ddot{\theta}_4(\theta_i, \dot{\theta}_i)} \\
\ddot{\theta}_3 &= \ddot{\theta}_3(\theta_i, \dot{\theta}_i, \ddot{\theta}_4) \\
&= \boxed{\ddot{\theta}_3(\theta_i, \dot{\theta}_i)} \\
\ddot{\theta}_2 &= \ddot{\theta}_2(\theta_i, \dot{\theta}_i, \ddot{\theta}_3, \ddot{\theta}_4) \\
&= \boxed{\ddot{\theta}_2(\theta_i, \dot{\theta}_i)} \\
\ddot{\theta}_1 &= \ddot{\theta}_1(\theta_i, \dot{\theta}_i, \ddot{\theta}_2, \ddot{\theta}_3, \ddot{\theta}_4) \\
&= \boxed{\ddot{\theta}_1(\theta_i, \dot{\theta}_i)}
\end{aligned}
$$

Figure 3: Backward Substitution for 4 links. The substitutions start from the last equation and then successively work back to the first equation. The order of substitutions per iteration are irrelevant. The result is all accelerations being expressed solely in terms of generalized coordinates and velocities

Note, the forward and backward substitutions can be computed symbolically, MATLAB was utilized for this. Now that all of the equations of motions have the generalized accelerations isolated, the system can be cast as a set of first order differential equations expressed as

$$
\begin{aligned}
\frac{d\theta_1}{dt} &= \dot{\theta}_1 \\
\frac{d\dot{\theta}_1}{dt} &= \ddot{\theta}_1(\theta_i, \dot{\theta}_i) \\
&\vdots \\
\frac{d\theta_N}{dt} &= \dot{\theta}_N \\
\frac{d\dot{\theta}_N}{dt} &= \ddot{\theta}_N(\theta_i, \dot{\theta}_i)
\end{aligned}
$$

From this, Runge-Kutta Methods can be employed to solve this system numerically.

1.3 Closing Remarks

Although this system was assumed to be 2D, the analysis could be generalized to 3D as well, this be used to capture the dynamics of a general N-link kinematic chain (e.g. robot manipulator). we assumed that the links were mass less, however we typically deal with rigid bodies with mass in real life. This analysis could be generalized to rigid bodies by reformulating the Lagrangian using the center of mass of each link. Lastly, instead of using the Euler-Lagrange Equations, we could've instead used the Newton-Euler Formulation, the equations of motion would be identical.